

# Several questions and hypotheses concerning the limit polynomials for Chacon<sub>(3)</sub> transformation

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We study the weak closure  $\mathcal{L} = \text{WCl}(\{\hat{T}^k\})$  of powers of non-singular Chacon transformation with 2-cuts. It is still an open question does  $\mathcal{L}$  contain any Markov operator except an orthogonal projector to the constants  $\Theta$  and some polynomials  $P(\hat{T})$ ? In this paper we calculate a particular set of limit polynomials

$$P_m(\hat{T}) = \lim_{n \rightarrow \infty} \hat{T}^{-mh_n}, \quad m \in \mathbb{Z},$$

where  $h_n = (3^n - 1)/2$  are the sequence of heights of towers in a standard rank one representation of the Chacon map. We show that for any  $d \geq 2$  the family of limit polynomials contains infinitely many polynomials of degree  $d$ . We also formulate hypotheses and open questions concerning the sequence  $P_m$  and the entire set  $\mathcal{L}$ .

## I. INTRODUCTION

Chacon transformation in terms of symbolic dynamics can be defined as a substitution system over the finite alphabet  $\mathbb{A} = \{0, 1\}$  via a pair of substitution rules

$$0 \mapsto 0010, \quad 1 \mapsto 1.$$

Starting with an initial word  $w_0 = 0$  and applying the substitution transform we construct the sequence  $w_n$ ,

$$w_0 = 0$$

$$w_1 = 0010$$

$$w_2 = 0010001010010$$

$$w_3 = 0010001010010001000101001010010001010010$$

...

and then define an infinite word  $w_\infty$  such that each  $w_n$  is a prefix of  $w_\infty$ . Further, considering the closure  $X$  of all shifts of  $w_\infty$  in the space  $\mathbb{A}^\infty$  endowed with the Tikhonov topology we come to a topological dynamical system  $(S, X, \mathcal{B})$ , where  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel sets and  $T$  is the shift transformation,

$$T: \dots, x_0, x_1, \dots, x_j, \dots \mapsto \dots, x_1, x_2, \dots, x_{j+1}, \dots$$

Let us consider a natural invariant measure  $\mu$  on the measurable space  $(X, \mathcal{B})$  defined as follows. For a finite word  $w$  let  $\mu([w])$  be the empirical probability of observing  $w$  in  $w_\infty$ , where  $[w]$  is the set encoded by  $w$ :

$$[w] := \{x \in X: x_0 = w(0), \dots, x_{\ell-1} = w(\ell-1)\},$$

$\ell = |w|$  is the length of  $w$  and  $w(j)$  denotes the letter at position  $j$  in  $w$ .

**Definition 1.** The map  $T$  considered as a measure-preserving invertible transformation of the probability space  $(X, \mathcal{B}, \mu)$  is called *non-singular Chacon transformation with 2-cuts* or *Chacon<sub>(3)</sub> transformation* (see [Cha69, Fri70]).

Transformation  $T$  has an interesting combination of ergodic properties. It is known to be weakly mixing and power weakly mixing [Dan04], but not strongly mixing [Cha69]. It has trivial centralizer [dJ78] and minimal self-joinings [dJRS80]. It is also known that the spectral measure  $\sigma$  of Chacon transformation  $T$  is singular and its convolutions satisfy the following condition of pairwise singularity [PR],

$$\sigma \perp \sigma * \sigma,$$

$$\sigma * \sigma \perp \sigma * \sigma * \sigma,$$

...

$$\sigma^{*k} \perp \sigma^{*\ell} \quad \text{for any } k \neq \ell.$$

The study of convolutions of the spectral type measure  $\sigma$  goes back to the Kolmogorov's question concerning the hypothetical group property of spectrum: *is it true that  $\sigma * \sigma \ll \sigma$ ?* This property holds for the discrete part of spectrum, but it is false for the singular component. Moreover, now we know many examples of ergodic transformations  $T$  such that  $\sigma * \sigma \perp \sigma$  (see [Ose69, Ste87, Goo99, dJL92]).

For a survey of problems in modern spectral theory of dynamical systems the reader can refer to [Lem09] and [KT07].

**Definition 2.** We say that a map  $T$  is *mixing* if

$$\mu(T^k A \cap B) \rightarrow \mu(A) \mu(B) \quad \text{as } k \rightarrow \infty,$$

for any measurable sets  $A$  and  $B$ , and we call  $T$  *weakly mixing* if the convergence holds for a subsequence  $k_j$ .

Both mixing and weak mixing properties can be described in spectral terms.

**Definition 3.** Let  $\hat{T}$  be the unitary *Koopman operator*, associated with  $T$  and acting in the separable Hilbert space  $H = L^2(X, \mu)$  by the following rule

$$\hat{T}: f(x) \mapsto f(Tx).$$

A sequence of bounded linear operators  $\mathcal{A}_j: H \rightarrow H$  in a Hilbert space  $H$  converges weakly to  $\mathcal{A}$  if for any  $f, g \in H$

$$\langle \mathcal{A}_j f, g \rangle \rightarrow \langle \mathcal{A} f, g \rangle, \quad j \rightarrow \infty.$$

Let  $\Theta$  denote the orto-projector to constants,

$$(\Theta f)(x) \equiv \int_X f(z) d\mu(z).$$

A transformation  $T$  is weakly mixing if and only if

$$T^{k_j} \rightarrow \Theta$$

for some subsequence  $k_j$ . It means that  $\Theta$  is in the weak closure  $\mathcal{L} = \text{WCl}(\{\hat{T}^k\})$  of powers  $\hat{T}^k$ .

## II. LIMIT POLYNOMIALS

In our investigation [PR] to prove the pairwise singularity of the convolutions  $\sigma^{*k}$  we used the following observation.

**Lemma 4.** *In the weak close of powers  $\mathcal{L} = \text{WCl}(\{\hat{T}^k\})$  for Chacon transformation  $T$  one can find an infinite family of non-trivial square polynomials*

$$Q_m(\hat{T}) = \frac{(3^s - 1)\mathbb{I} + 2(3^s + 1)\hat{T} + (3^s - 1)\hat{T}^2}{4 \cdot 3^s},$$

for  $m = 3^s + 1$  and, moreover,

$$Q_m(\hat{T}) = \lim_{n \rightarrow \infty} \hat{T}^{mh_n - l_s},$$

where  $l_s = (3^s - 1)/2$  and  $\mathbb{I}$  is the identity operator.

In order to understand this phenomem let us consider a simpler case.

**Lemma 5.** *There exists a sequence  $k_j \rightarrow \infty$  such that*

$$\hat{T}^{k_j} \rightarrow \frac{\mathbb{I} + \hat{T}}{2}.$$

*Proof.* Another way to define Chacon transformation as a measure-preserving transformation is to use the concept of rank one transformation.

**Definition 6.** Let  $T$  be a measure-preserving transformation of a probability space  $(X, \mathcal{B}, \mu)$ . Then  $T$  is called *rank one transformation* if there exists a sequence of Rokhlin tower partitions

$$\xi_j = \{B_j, TB_j, T^2B_j, \dots, T^{h_n-1}B_j, E_j\}$$

of the phase space such that  $\mu(E_j) \rightarrow 0$  and for any measurable set  $A$  one can fins  $\xi_j$ -measurable sets  $A_j$  approximating  $A$ :  $\mu(A_j \Delta A) \rightarrow 0$  as  $j \rightarrow \infty$ .

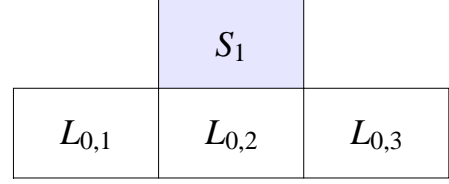


FIG. 1: Chacon<sub>(3)</sub> transformation: several steps in the cutting-and-stacking construction:  $n = 1$

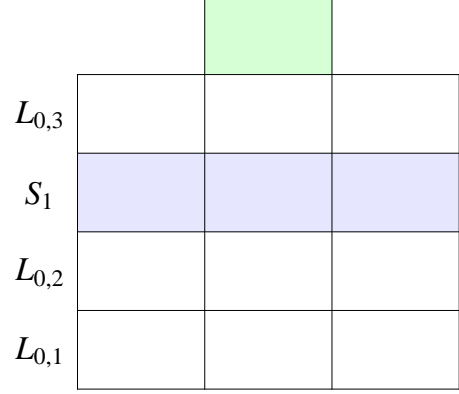


FIG. 2: Cutting-and-stacking construction:  $n = 2$

In fact, Chacon transformation is rank one and can be constructed using so-called cutting-and-stacking construction.

**Construction 7.** We start with a unit segment  $[0, 1]$  interpreted as a Rokhlin tower  $U_0$  of height  $h_0 = 1$ . Then we cut this segment twice, in three equal parts

$$L_{1,0} = [0, 1/3), \quad L_{1,1} = [1/3, 2/3), \quad L_{1,2} = [2/3, 1],$$

and add one additional “level”, a segment  $S_1$  of length  $1/3$  which is drawn above the middle part  $[1/3, 2/3)$  (see fig. 1),

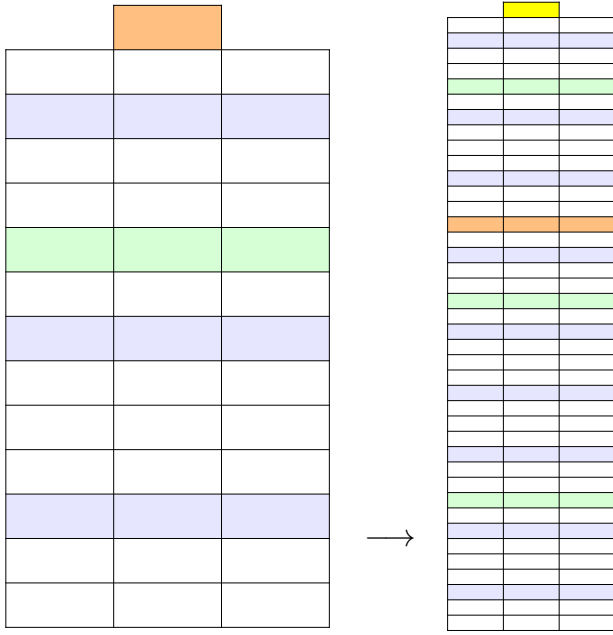
$$\begin{array}{c} S_1 \\ L_{1,0} \quad L_{1,1} \quad L_{1,2} \end{array}$$

Now we stack all these segments in the natural order:  $L_{1,0} L_{1,1} S_1 L_{1,2}$  and we get the next Rokhlin tower  $U_1$  of height  $h_1 = 4$  (see fig. 2). In other words, we assume that

$$L_{1,0} \xrightarrow{T} L_{1,1} \xrightarrow{T} S_1 \xrightarrow{T} L_{1,2},$$

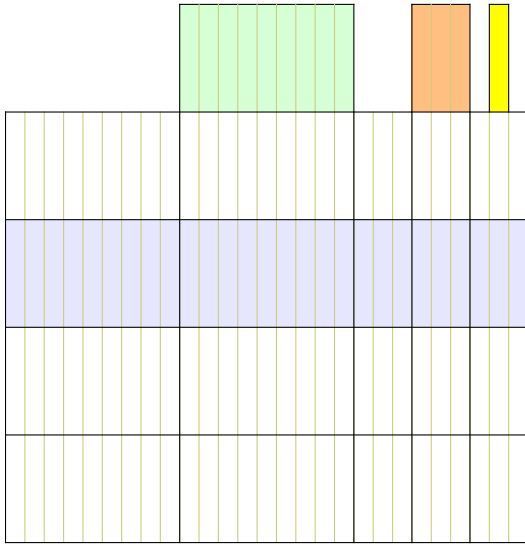
and  $T$  will be defined on  $L_{1,2}$  on the next steps of the construction. We repeat the same procedure with the new tower: we cat it in three equal columns, put one additional level to the top of the middle column and stack together (fig. 1–1).

At each step of the construction we have a Rokhlin tower  $U_n$  of height  $h_n = (3^n - 1)/2$ . It can be easily checked that this sequence serves as an approximating

FIG. 3: Cutting-and-stacking construction:  $n = 3$  and  $n = 4$ 

sequence of Rokhlin towers in the definition of rank one transformation.

Note that if we draw all the additional level above the corresponding subcolumns without restacking the tower  $U_n$  at the step  $n$  we come to the following representation of the Chacon map (see fig. 4).

FIG. 4: Chacon<sub>(3)</sub> transformation without restacking

**Construction 8.** Let us consider the compact group of 3-adic integers  $\Gamma = \mathbb{Z}_{(3)}$ . We associate  $\Gamma$  with the set of one-sided 3-adic sequences

$$y = (y_1, y_2, \dots, y_k, \dots), \quad y_k \in \{0, 1, 2\}.$$

As a measure space  $\Gamma$  is isomorphic to the unit segment  $[0, 1]$  by the mapping

$$y \mapsto \sum_{k=1}^{\infty} \frac{y_k}{3^k}.$$

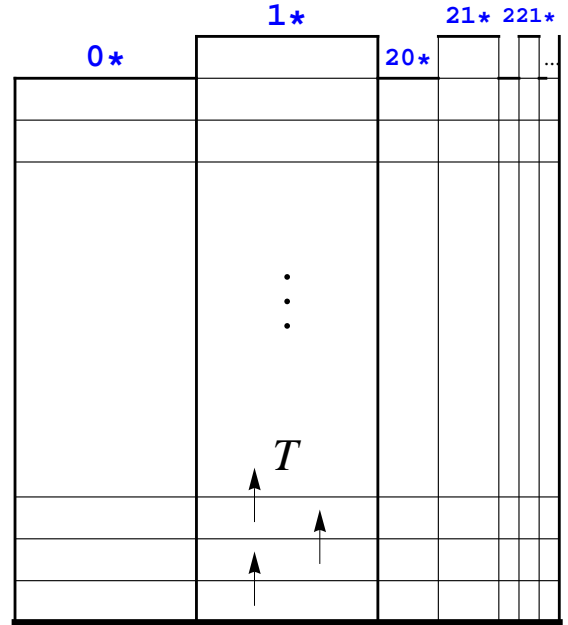
It follows easily from the cutting-and-stacking construction that Chacon map  $T$  is the integral transformation over the adding machine transformation

$$S: \Gamma \rightarrow \Gamma: y \rightarrow y + 1$$

acting on the base level of the tower  $U_n$  identified with  $\Gamma$  with the ceiling function  $r_n(y) = h_n + \phi_0(y)$  (see fig. 5),

$$\phi_0(y) = \begin{cases} 0, & \text{if } y = 22 \dots 20* \\ 1, & \text{if } y = 22 \dots 21* \end{cases}$$

where  $*$  indicates any symbol in alphabet  $\{0, 1, 2\}$  if put inside a block, and any infinite sequence of symbols if it ends the block.

FIG. 5: Chacon<sub>(3)</sub> transformation: cutting-and-stacking construction and cocycle  $\phi_0(y)$ .

Now we are ready to finish the proof of lemma 5. It can be easily checked that any measurable function  $f \in L^2(X, \mu)$  is approximated by functions constant on levels of a tower in the rank one representation. So, assume that  $f$  is constant on the levels of  $U_n$ . Let us partition  $U_n$  into sets  $U_n^{(0)}$  and  $U_n^{(1)}$  according to the value of the cocycle  $\phi_0(y)$ , where  $y$  is considered as a point in the base of  $U_n$ . We see that

$$f(T^{h_n} x) = f(x), \quad \text{if } x \in U_n^{(0)}$$

and

$$f(T^{h_n}x) = f(T^{-1}x), \quad \text{if } x \in U_n^{(1)}$$

for all points  $x \in U_n$  except the first level  $B_n$  of the tower  $U_n$  (observe that  $\mu(B_n) \rightarrow 0$ ). Thus,

$$\hat{T}^{h_n} \rightarrow \frac{\mathbb{I} + \hat{T}^{-1}}{2},$$

since  $\mu(U_n^{(0)}) = \mu(U_n^{(1)})$ , and applying conjugation we complete the proof.  $\square$

Analyzing the effects used in the proof we see that lemma 5 can be easily extended in the following way. Given  $m \in \mathbb{N}$  let us consider the sum

$$\phi_0^{(m)} = \phi_0(y) + \phi_0(Sy) + \dots \phi_0(S^{m-1}y)$$

and define the corresponding distribution  $\rho_m$  of the values of  $\phi_0^{(m)}$ . Actually  $\rho_m$  is the measure on  $\mathbb{Z}$  with a finite support which is the image by  $\phi_0^{(m)}$  of the Haar probability measure on  $\Gamma$ .

**Lemma 9.** *For any  $m \in \mathbb{N}$  the sequence  $\hat{T}^{-mh_n}$  converges weakly to a polynomial depending on  $\hat{T}$ , and*

$$P_m(\hat{T}) := \lim_{n \rightarrow \infty} \hat{T}^{-mh_n} = \int_{\mathbb{Z}} \hat{T}^k d\rho_m(k).$$

The scheme of the proof can be found in [PR], and the idea can be explained as follows. Passing the tower  $U_n$   $m$  times we count (in addition to  $mh_n$ ) the values of the cocycle  $\phi_0(y)$  at the points

$$\phi_0(y), \quad \phi_0(Sy), \quad \dots \phi_0(S^{m-1}y).$$

Let us consider several first polynomials  $P_n(\hat{T})$ :

$$\begin{aligned} P_1(\hat{T}) &= \frac{1}{2}(\mathbb{I} + \hat{T}) \\ P_2(\hat{T}) &= \frac{1}{6}(\mathbb{I} + 4\hat{T} + \hat{T}^2) \\ P_3(\hat{T}) &= \frac{1}{2}(\hat{T} + \hat{T}^2) \\ P_4(\hat{T}) &= \frac{1}{9}(2\hat{T} + 5\hat{T}^2 + 2\hat{T}^3) \\ P_5(\hat{T}) &= \frac{1}{18}(\hat{T} + 8\hat{T}^2 + 8\hat{T}^3 + \hat{T}^4) \end{aligned}$$

Since the weak closure  $\text{WCl}(\{\hat{T}^j\})$  is invariant under multiplication by  $\hat{T}^s$  for any  $s \in \mathbb{Z}$  we can reduce the polynomials  $P_m(\hat{T})$  by the smallest power  $l_m$  of  $\hat{T}$  in  $P_m(\hat{T})$ . Set

$$\tilde{P}_m(z) = z^{-l_m} \cdot P_m(z).$$

Let us represent  $\tilde{P}_m(z)$  in the form

$$\tilde{P}_m(z) = a_{m,0} + a_{m,1}z + \dots + a_{m,d(m)}z^{d(m)},$$

where  $d(m) = \deg \tilde{P}_m(z)$ .

**Lemma 10.** *The coefficients  $a_{m,j} \in \mathbb{Q}$  satisfy the following Markov property*

$$\sum_{j=0}^{d(m)} a_{m,j} = 1, \quad \text{and} \quad a_{m,j} \geq 0.$$

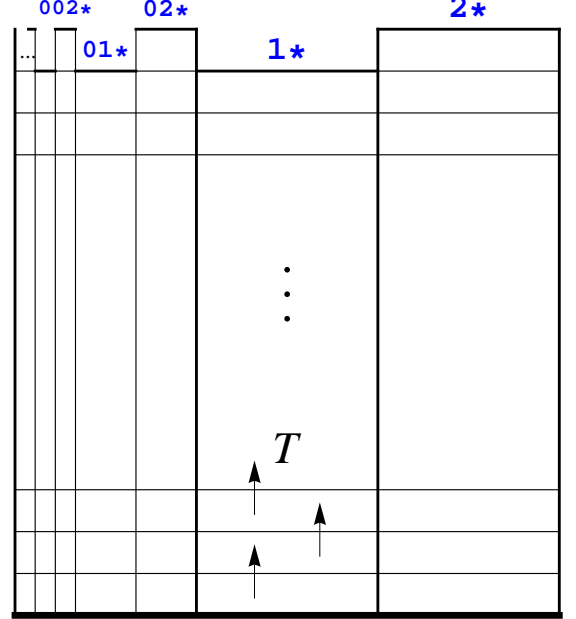


FIG. 6: Chacon<sub>(3)</sub> transformation after the coordinate change  $y \mapsto y + 1$  and cocycle  $\phi(y)$ .

In table 1 of the Appendix we list the first 365 polynomials  $\tilde{P}_m(z)$ .

Let us discuss several remarks explaining the structure of this table. First, for simplicity of calculations we apply the transform  $y \mapsto y + 1$  to the base of the tower  $U_n$  and consider the following cocycle  $\phi(y)$  instead of  $\phi_0(y)$  (see fig. 6),

$$\phi(y) = \begin{cases} 0, & \text{if } y = 00 \dots 01* \\ 1, & \text{if } y = 00 \dots 02* \end{cases}$$

The function  $\phi(y)$  is more convenient for calculation of the iterates  $\phi(S^k y)$ .

**Lemma 11.** *For any power  $3^\ell$  of three we have*

$$\phi^{(3^\ell)}(y) = \begin{cases} 0, & \text{if } y = *^\ell(0)1* \\ 1, & \text{if } y = *^\ell(0)1* \end{cases}$$

where the notation  $(0)$  is used for any sequence of zeros (including empty sequence), and  $*^\ell$  denotes an arbitrary word of length  $\ell$ . An equivalent way how we can state this property of the cocycle is to say that

$$\phi^{(3^\ell)}(y) = \phi(Ay),$$

where  $A$  is the non-invertible left shift:

$$A(y_1 y_2 \dots y_k \dots) = y_2 y_3 \dots y_{k+1} \dots$$

It follows immediately from this lemma that polynomials  $\tilde{P}_m(z)$  repeat after multiplication by 3,

$$P_{3m}(z) = P_m(z).$$

**Theorem 12.** *For any  $d \in \mathbb{N}$  the family  $\{\tilde{P}_m\}$  contains infinitely many polynomials of degree  $d$ .*

*Proof.* The theorem is based on the following observation. If we consider functions  $\phi^{(m)}(y)$  and  $\phi(S^k y)$  as random variables defined on  $\Gamma$ , then  $\phi^{(m)}(y)$  and  $\phi(y)$  are almost independent. Thus, extending the proof of lemma 4 we see that configurations

$$m(\ell_1, \dots, \ell_{d-1}) = 10^{\ell_1} 10^{\ell_2} 100 \dots 0^{\ell_{d-1}} 1_3$$

generates for sufficiently big  $\ell_j$  polynomials  $P_{(\ell_j)}(z)$  of degree  $d$  such that

$$\lim_{\ell_j \rightarrow \infty} P_{(\ell_j)}(\hat{T}) = \frac{1}{2^d} (\mathbb{I} + \hat{T})^d.$$

□

To illustrate the construction used in the proof let us consider configuration

$$m(\ell_1, \ell_2) = 10^{\ell_1} 10^{\ell_2} 1_3 = \overbrace{100 \dots 0}^{\ell_1} \overbrace{1000 \dots 0}^{\ell_2} 1_3$$

Set

$$p^{-[i,j]} = p^{-i} + p^{-i-1} + \dots + p^{-j},$$

and notice that  $3^{-[1,\infty]} = 1/2$ .

**Lemma 13.**  $\mathcal{P}_{m(\ell_1, \ell_2)}$  is a self-reciprocal polynomial,

$$\mathcal{P}_{m(\ell_1, \ell_2)} = \gamma z^3 + (1/2 - \gamma) z^2 + (1/2 - \gamma) z + \gamma,$$

where

$$\gamma = 3^{-[1, \ell_1]} 3^{-[1, \ell_2]} + 3^{-[1, \ell_1]} 3^{-(\ell_2+1)} + 3^{-(\ell_1+1)} 3^{-[1, \ell_2]}.$$

*Proof.* The proof of this lemma is very close to that of lemma 4. □

In the next section we formulate a set of hypotheses concerning the properties of the limit polynomials  $\tilde{P}_m(z)$ . In hypothesis 1 we conjecture that all polynomials  $\tilde{P}_m(z)$  are self-reciprocal, i.e. they have coefficients symmetric under the transform  $k \mapsto d - 1$ ,  $d = \deg \tilde{P}_m$ ,

$$\tilde{P}_m(z) = \sum_{j=0}^{d(m)} a_{m,j} z^j, \quad a_{m,j} = a_{m, d(m)-j}.$$

In other words, the sequence  $a_{m,j}$  is symmetric.

**Lemma 14.** *If hypothesis 1 holds then  $d(m) \in 2\mathbb{Z} + 1$  implies that  $(-1)$  is a root of  $\tilde{P}_m$ . In particular, any polynomial  $\tilde{P}_m$  of odd degree is factorized,*

$$\mathcal{P}_m(z) = (z + 1) R_m(z).$$

The proof is a simple calculation. Nevertheless, we can ask a question: is it the only way to factorize  $\tilde{P}_m$ ?

**Theorem 15.** *The family of limit polynomials  $\mathcal{P}_m(z)$  contains infinitely many cubic polynomials for which  $R_m(z) = (z + 1)^{-1} \tilde{P}_m(z)$  are irreducible.*

*Proof.* Indeed, consider cubic polynomials given by configurations  $10^{\ell_1} 10^{\ell_2} 1$  with  $\ell_1 = \ell_2$  (see table 2 of the Appendix). With a simplified notation  $\ell = \ell_1$  we have

$$\begin{aligned} \mathcal{P}_{m(\ell, \ell)}(z) &= \\ &= \frac{(3a^2 + 2a)(z^3 + 1) + (3^{2\ell+1} - 3a^2 - 2a)(z^2 + z)}{2 \cdot 3^{2\ell+1}}, \end{aligned}$$

where

$$3^{-[1, \ell]} = \frac{1}{3} + \dots + \frac{1}{3^\ell} = \frac{a}{3^\ell}, \quad \gcd(a, 3) = 1.$$

Next, let us apply the transform  $z = -1 + w$  to  $\mathcal{P}_{m(\ell, \ell)}$ . We get a new polynomial

$$P^*(w) = (3a^2 + 2a)w^3 + (3^{2\ell+1} - 8a - 12a^2)(w - 1).$$

What are the common divisors of

$$X = 3a^2 + 2a \quad \text{and} \quad Y = 3^{2\ell+1} - 8a - 12a^2?$$

We have

$$Y + 4X = 3^{2\ell+1}$$

and, at the same time,

$$X = a(3a + 2)$$

is factorized in two numbers, both are relatively prime to 3. Thus, taking any prime divisor of  $Y$  and applying Eisenstein's criterion we see that  $P^*(w)$  is irreducible over  $\mathbb{Q}$ . □

**Lemma 16** (Eisenstein's criterion). *Consider a polynomial  $P \in \mathbb{Q}[z]$ ,*

$$P(z) = a_n z^n + \dots + a_1 z + a_0,$$

*and suppose that there exists a prime number  $p$  such that*

$$\begin{aligned} p &\nmid a_n, & p^2 &\nmid a_0, \\ p &\mid a_j \quad \text{for } j = 0, 1, \dots, n-1. \end{aligned}$$

*Then  $P(z)$  is irreducible over  $\mathbb{Q}$ .*

It is interesting to remark that the quadratic polynomials given in lemma 4 are factorized over  $\mathbb{Q}$ , thus, to see that there exists irreducible polynomial  $\tilde{P}_m(z)$  we have to consider a particular example:

$$\tilde{P}_2(z) = \frac{1}{6}(z^2 + 4z + 1).$$

Letting  $z = -1 + w$  we get a polynomial

$$P^*(w) = z^2 + 2z - 2.$$

We can apply Eisenstein's criterion to  $P^*$ , since 2 divides all the coefficient except the coefficient in  $z^2$ , and 4 do not divide  $-2$ .

### III. QUESTIONS AND HYPOTHESES

**Hypothesis 1.** The limit polynomials  $\tilde{P}_m(z)$  are *self-reciprocal*, that is

$$\tilde{P}_m(z) = \sum_{k=0}^{d(m)} a_k z^k, \quad a_k = a_{d(m)-k}.$$

*Corollary.* If hypothesis 1 is true then  $-1$  is a root of a polynomial  $\tilde{P}_m(z)$ , whenever  $d(m) \in 2\mathbb{Z} + 1$ .

We have to mention that most questions below presume or at least require hypothesis 1 for a particular  $m$ .

**Definition 17.** Consider two configurations of the same length

$$c_1 c_2 \dots c_N \quad \text{and} \quad c'_1 c'_2 \dots c'_N,$$

with  $c_j, c'_j \in \{0, 1, 2\}$ . We say that  $m'$  is *conjugate* to  $m$  and write  $m' = m^*$  if  $c'_j = c_{N+1-j}$ .

**Hypothesis 2.** The polynomials  $\tilde{P}_m(z)$  and  $\tilde{P}_{m^*}(z)$  coincide for any pair of conjugate configurations  $m$  and  $m^*$ .

It can be observed from table 1 that some polynomials coincide even for non-conjugate configurations, for example, for  $m = 10 = 101_3$  and  $m' = 26 = 222_3$ .

**Question 3.** Which pairs of polynomials  $\tilde{P}_m(z)$  and  $\tilde{P}_{m'}(z)$  coincide?

Let  $|m|_3$  be the length of the 3-adic expansion of  $m$  if  $3 \nmid m$ , and  $|m|_3 = |3^{-1}m|_3$  otherwise.

**Hypothesis 4.**  $P_m^{\mathbb{Z}}(z) = 2 \cdot 3^{|m|_3} \cdot \tilde{P}_m(z)$  is a polynomial with integer coefficients,

$$P_m^{\mathbb{Z}} = b_{m,0} + b_{m,1}z + \dots + b_{m,d(m)}z^d.$$

The greatest common divisor of  $b_{m,j}$  is 1 or 2.

This is a well-known fact that the set of all weak limits of powers  $\mathcal{L}$  is a semigroup. Thus, it is a natural question: can we get a polynomial  $P_m(z)$  as a product of two different elements of  $\mathcal{L}$ ?

**Hypothesis 5.** The polynomial  $\tilde{P}_m(z)$  has two or more factors which are not  $(z+1)$  if and only if (see table 3)

$$|m|_3 \in 2\mathbb{Z} \quad \text{and} \quad m = m^*.$$

For example, the polynomial

$$\tilde{P}_{68}(z) = \frac{1}{81}(3 + 5z + z^2)(1 + 5z + 3z^2)$$

corresponds to a symmetric configuration  $68 = 2112_3$ .

In particular, if hypothesis 5 is true then the roots  $r_j$  of a polynomial  $P_m^{\mathbb{Z}}(z)$  starting with  $z^d + \dots$  are algebraic integers.

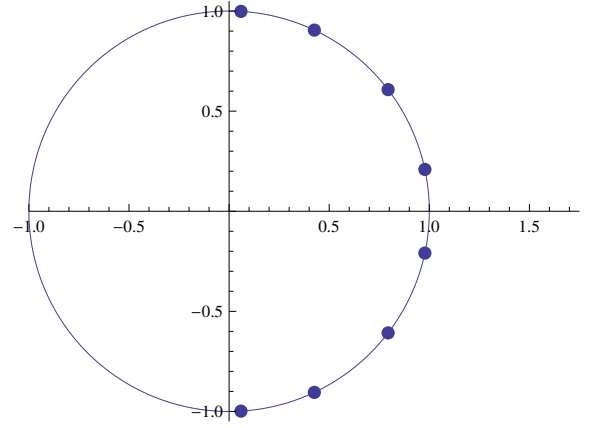


FIG. 7: Roots of the polynomials  $Q_{1094}$ .

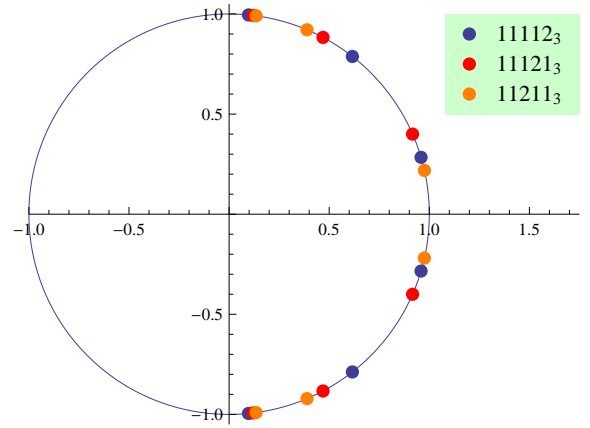


FIG. 8: Roots of the polynomials  $Q_{122}$ ,  $Q_{124}$  and  $Q_{130}$ .

**Hypothesis 6.** All roots of any  $\tilde{P}_m(z)$  are real numbers (*Lee–Yang property*).

*Remark 18.* It follows directly from hypothesis 1 as well as the definition of the polynomials  $\tilde{P}_m(z)$  that the roots of  $\tilde{P}_m(z)$  must be negative, and they appear in pairs:  $r$  and  $r^{-1}$ .

Now, if we assume hypotheses 1, 5 and 6 then applying to our polynomials the transformation

$$z = \kappa_1(z) = i \frac{z-1}{z+1}$$

mapping  $\mathbb{R}$  to the unit circle in the complex plane, we can define the *dual polynomials*

$$Q_m(w) = \tilde{P}_m(\kappa_1(z)).$$

**Hypothesis 7.** The polynomials  $Q_m(w)$  are self-reciprocal polynomials having all roots  $\lambda_j$  on the unit circle and in the right-half plane:

$$|\lambda_j| = 1, \quad \text{Re } \lambda_j > 0.$$

Let us consider, for example, the polynomials

$$P_{122}^{\mathbb{Z}}(z) = z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1$$

$$P_{124}^{\mathbb{Z}}(z) = z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1$$

$$P_{130}^{\mathbb{Z}}(z) = z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1,$$

corresponding to the configuration

$$122 = 11112_3, \quad 124 = 11121_3, \quad 130 = 11211_3.$$

The dual polynomials  $Q_m(w)$  are

$$Q_{122}(w) = \frac{-2i}{486} (35w^6 - 117w^5 + 209w^4 - 250w^3 + 209w^2 - 117w + 35)$$

$$Q_{124}(w) = \frac{-i}{486} (77w^6 - 232w^5 + 415w^4 - 496w^3 + 415w^2 - 232w + 77)$$

$$Q_{130}(w) = \frac{-2i}{486} (39w^6 - 117w^5 + 205w^4 - 250w^3 + 205w^2 - 117w + 39),$$

and the root of these polynomials are shown on fig. 8.

**Question 8.** What is the asymptotic behaviour of the distributions  $\rho_m$ ?

*Remark.* In the proof of theorem 12 we consider, for a given degree  $d$ , a set of polynomials corresponding to configurations

$$m = 1 \ 0^{\ell_1} \ 10^{\ell_2} \ 1 \dots 0^{\ell_{d-1}} \ 1_3,$$

where ones are separated by long sequences of zeroes. These configurations generate sums  $\phi^{(m)}$  which are reduced to sums of  $d$  almost independent random variables, and, in particular,

$$\tilde{P}_m(z) \rightarrow \frac{1}{2^d} (1+z)^d, \quad \ell_j \rightarrow \infty.$$

Thus, it is easy to see that the corresponding distributions  $\rho_m$  converge to the binomial distribution.

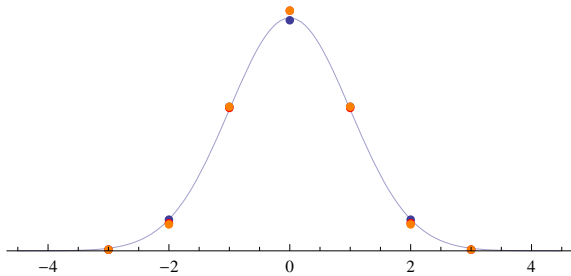


FIG. 9: The distributions  $\rho_{122}$ ,  $\rho_{124}$  and  $\rho_{130}$  and the normal distribution.

**Question 9.** Is it true that the distributions  $\rho_m$ , centered and scaled, converge to the normal distribution as  $d(m) \rightarrow \infty$  independently on the structure of  $\tilde{P}_m(z)$ ? (see fig. 9)

**Hypothesis 10.** The first polynomials  $\tilde{P}_m(z)$  of degree  $d$  is observed at

$$m = \frac{3^{d-1} + 1}{2}. \quad (1)$$

**Hypothesis 11.** Consider a subsequence of polynomials described in hypothesis 10. If  $m$  is given by formula (1) and  $m \in 2\mathbb{Z}$  then the corresponding polynomials  $P_m^{\mathbb{Z}}(z)$  are irreducible monic self-reciprocal polynomials. If  $m$  is odd then the same is true for  $(z+1)^{-1}P_m^{\mathbb{Z}}(z)$ . The roots  $r_j$  of  $P_m^{\mathbb{Z}}(z)$  are algebraic integers, moreover,  $r_j \in \mathbb{R}$ .

*Remark 19.* In particular, if hypothesis 10 is true then the following estimate holds

$$d(m) \leq 1 + \log_3(2m - 1).$$

**Question 12.** How  $d(m)$  depends on  $m$ ?

**Question 13.** Is it true that no one polynomial  $\tilde{P}_m(z)$  divides another polynomials in this sequence, and any  $\tilde{P}_m(z)$  is never a product of different polynomials  $\tilde{P}_{m'_k}(z)$  of smaller degree?

**Question 14.** Is it true that any operator  $\tilde{P}_m(\mathbf{T})$  is not a product of different operators  $A_j \in \mathcal{L}$  in the weak closure of powers of Chacon transformation  $\hat{T}$ ?

**Question 15.** Can we find  $\tilde{P}_m(z)$  which is an isolated point in the semigroup generated by all  $\{\tilde{P}_{m'}\}$ , and can we find  $\tilde{P}_m(\mathbf{T})$  which is an isolated point in  $\mathcal{L}$ ?

**Question 16.** Is it true that the set  $\mathcal{L}$  does not contain operators given by series

$$\sum_{j \in \mathbb{Z}} a_j \hat{T}^j,$$

where infinitely many  $a_j \neq 0$ ? Is it possible to find among elements  $V \in \mathcal{L}$  operators of the form

$$V = \kappa \Theta + \sum_j a_j \hat{T}^j, \quad \kappa \neq 0?$$

The following well-known question still has no answer as well.

**Question 17.** Is Chacon<sub>(3)</sub> transformation is  $\kappa$ -mixing, which means that there exists  $V \in \mathcal{L}$  such that

$$V = \kappa \Theta + V_2, \quad \kappa \neq 0?$$

**Hypothesis 18.** There exists  $\varepsilon_0 > 0$  such that among the polynomials  $P_m(\hat{T})$  and in the set  $\mathcal{L}$  there is no polynomials

$$\sum_{j=0}^d a_j \hat{T}^j, \quad a_j > 0,$$

having the property

$$\left| \frac{a_{j+1}}{a_j} - 1 \right| < \varepsilon_0.$$

**Question 19.** Can we find  $\tilde{P}_m(z)$  which is an isolated point in the semigroup generated by all  $\{\tilde{P}_{m'}\}$ , and can we find  $\tilde{P}_m(\mathbf{T})$  which is an isolated point in  $\mathcal{L}$ ?

**Question 20.** How we can describe the entire set  $\mathcal{L}$  for Chacon<sub>(3)</sub> transformation?

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#### V. APPENDIX: THE LIMIT POLYNOMIALS

**Table 1.** First 122 limit polynomials  $\tilde{P}_m(z)$

The columns of this table indicate: the number  $m$ , 3-adic expansion of  $m$  (configuration), and the polynomial  $\tilde{P}_m(z)$ . We mark by \* the indexes corresponding to configurations  $111 \dots 12_3$ . We skip symmetrical configurations like  $112_3 \sim 211_3$  following Hypothesis 2 which is true in this interval.

Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
1*	1 <sub>3</sub>	$\tilde{P}_1(z) = \tilde{P}_3(z) = \tilde{P}_9(z) = \dots = \frac{1}{2}(1+z)$
2*	2 <sub>3</sub>	$\tilde{P}_2(z) = \tilde{P}_6(z) = \dots = \frac{1}{6}(1+4z+z^2)$
4	11 <sub>3</sub>	$\tilde{P}_4(z) = \frac{1}{9}(2z^2+5z+2)$
5*	12 <sub>3</sub>	$\tilde{P}_5(z) = \frac{1}{18}(z^3+8z^2+8z+1)$
8	22 <sub>3</sub>	$\tilde{P}_8(z) = \frac{1}{9}(2z^2+5z+2)$
10	101 <sub>3</sub>	$\tilde{P}_{10}(z) = \frac{1}{54}(13z^2+28z+13)$
11	102 <sub>3</sub>	$\tilde{P}_{11}(z) = \frac{1}{54}(4z^3+23z^2+23z+4)$
13	111 <sub>3</sub>	$\tilde{P}_{13}(z) = \frac{1}{54}(5z^3+22z^2+22z+5)$
14*	112 <sub>3</sub>	$\tilde{P}_{14}(z) = \frac{1}{54}(z^4+13z^3+26z^2+13z+1)$
16	121 <sub>3</sub>	$\tilde{P}_{16}(z) = \frac{1}{54}(z^4+12z^3+28z^2+12z+1)$



Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
17	122 <sub>3</sub>	$\tilde{P}_{17}(z) = \frac{1}{54}(4z^3 + 23z^2 + 23z + 4)$
20	202 <sub>3</sub>	$\tilde{P}_{20}(z) = \frac{1}{54}(z^4 + 12z^3 + 28z^2 + 12z + 1)$
23	212 <sub>3</sub>	$\tilde{P}_{23}(z) = \frac{1}{54}(5z^3 + 22z^2 + 22z + 5)$
26	222 <sub>3</sub>	$\tilde{P}_{26}(z) = \frac{1}{54}(13z^2 + 28z + 13)$
28	1001 <sub>3</sub>	$\tilde{P}_{28}(z) = \frac{1}{81}(20z^2 + 41z + 20)$
29	1002 <sub>3</sub>	$\tilde{P}_{29}(z) = \frac{1}{162}(13z^3 + 68z^2 + 68z + 13)$
31	1011 <sub>3</sub>	$\tilde{P}_{31}(z) = \frac{1}{162}(17z^3 + 64z^2 + 64z + 17)$
32	1012 <sub>3</sub>	$\tilde{P}_{32}(z) = \frac{1}{81}(2z^4 + 20z^3 + 37z^2 + 20z + 2)$
34	1021 <sub>3</sub>	$\tilde{P}_{34}(z) = \frac{1}{162}(4z^4 + 39z^3 + 76z^2 + 39z + 4)$
35	1022 <sub>3</sub>	$\tilde{P}_{35}(z) = \frac{1}{162}(16z^3 + 65z^2 + 65z + 16)$
38	1102 <sub>3</sub>	$\tilde{P}_{38}(z) = \frac{1}{162}(5z^4 + 39z^3 + 74z^2 + 39z + 5)$
40	1111 <sub>3</sub>	$\tilde{P}_{40}(z) = \frac{1}{81}(3z^4 + 20z^3 + 35z^2 + 20z + 3)$
41*	1112 <sub>3</sub>	$\tilde{P}_{41}(z) = \frac{1}{162}(z^5 + 19z^4 + 61z^3 + 61z^2 + 19z + 1)$
43	1121 <sub>3</sub>	$\tilde{P}_{43}(z) = \frac{1}{162}(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1)$
44	1122 <sub>3</sub>	$\tilde{P}_{44}(z) = \frac{1}{81}(2z^4 + 20z^3 + 37z^2 + 20z + 2)$
47	1202 <sub>3</sub>	$\tilde{P}_{47}(z) = \frac{1}{162}(z^5 + 16z^4 + 64z^3 + 64z^2 + 16z + 1)$
50	1212 <sub>3</sub>	$\tilde{P}_{50}(z) = \frac{1}{162}(5z^4 + 39z^3 + 74z^2 + 39z + 5)$
52	1221 <sub>3</sub>	$\tilde{P}_{52}(z) = \frac{1}{81}(2z^4 + 18z^3 + 41z^2 + 18z + 2)$
53	1222 <sub>3</sub>	$\tilde{P}_{53}(z) = \frac{1}{162}(13z^3 + 68z^2 + 68z + 13)$

Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
56	2002 <sub>3</sub>	$\tilde{P}_{56}(z) = \frac{1}{81}(2z^4 + 18z^3 + 41z^2 + 18z + 2)$
59	2012 <sub>3</sub>	$\tilde{P}_{59}(z) = \frac{1}{162}(z^5 + 17z^4 + 63z^3 + 63z^2 + 17z + 1)$
62	2022 <sub>3</sub>	$\tilde{P}_{62}(z) = \frac{1}{162}(4z^4 + 39z^3 + 76z^2 + 39z + 4)$
68	2112 <sub>3</sub>	$\tilde{P}_{68}(z) = \frac{1}{81}(3z^4 + 20z^3 + 35z^2 + 20z + 3)$
71	2122 <sub>3</sub>	$\tilde{P}_{71}(z) = \frac{1}{162}(17z^3 + 64z^2 + 64z + 17)$
80	2222 <sub>3</sub>	$\tilde{P}_{80}(z) = \frac{1}{81}(20z^2 + 41z + 20)$
82	10001 <sub>3</sub>	$\tilde{P}_{82}(z) = \frac{1}{486}(121z^2 + 244z + 121)$
83	10002 <sub>3</sub>	$\tilde{P}_{83}(z) = \frac{1}{486}(40z^3 + 203z^2 + 203z + 40)$
85	10011 <sub>3</sub>	$\tilde{P}_{85}(z) = \frac{1}{486}(53z^3 + 190z^2 + 190z + 53)$
86	10012 <sub>3</sub>	$\tilde{P}_{86}(z) = \frac{1}{486}(13z^4 + 121z^3 + 218z^2 + 121z + 13)$
88	10021 <sub>3</sub>	$\tilde{P}_{88}(z) = \frac{1}{486}(13z^4 + 120z^3 + 220z^2 + 120z + 13)$
89	10022 <sub>3</sub>	$\tilde{P}_{89}(z) = \frac{1}{486}(52z^3 + 191z^2 + 191z + 52)$
91	10101 <sub>3</sub>	$\tilde{P}_{91}(z) = \frac{1}{486}(56z^3 + 187z^2 + 187z + 56)$
92	10102 <sub>3</sub>	$\tilde{P}_{92}(z) = \frac{1}{486}(17z^4 + 120z^3 + 212z^2 + 120z + 17)$
94	10111 <sub>3</sub>	$\tilde{P}_{94}(z) = \frac{1}{486}(21z^4 + 121z^3 + 202z^2 + 121z + 21)$
95	10112 <sub>3</sub>	$\tilde{P}_{95}(z) = \frac{1}{486}(4z^5 + 61z^4 + 178z^3 + 178z^2 + 61z + 4)$
97	10121 <sub>3</sub>	$\tilde{P}_{97}(z) = \frac{1}{486}(4z^5 + 56z^4 + 183z^3 + 183z^2 + 56z + 4)$
98	10122 <sub>3</sub>	$\tilde{P}_{98}(z) = \frac{1}{486}(16z^4 + 121z^3 + 212z^2 + 121z + 16)$
100	10201 <sub>3</sub>	$\tilde{P}_{100}(z) = \frac{1}{243}(8z^4 + 60z^3 + 107z^2 + 60z + 8)$

Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
101	10202 <sub>3</sub>	$\tilde{P}_{101}(z) = \frac{1}{486}(4z^5 + 55z^4 + 184z^3 + 184z^2 + 55z + 4)$
103	10211 <sub>3</sub>	$\tilde{P}_{103}(z) = \frac{1}{486}(4z^5 + 59z^4 + 180z^3 + 180z^2 + 59z + 4)$
104	10212 <sub>3</sub>	$\tilde{P}_{104}(z) = \frac{1}{243}(10z^4 + 60z^3 + 103z^2 + 60z + 10)$
106	10221 <sub>3</sub>	$\tilde{P}_{106}(z) = \frac{1}{486}(16z^4 + 117z^3 + 220z^2 + 117z + 16)$
107	10222 <sub>3</sub>	$\tilde{P}_{107}(z) = \frac{1}{486}(52z^3 + 191z^2 + 191z + 52)$
110	11002 <sub>3</sub>	$\tilde{P}_{110}(z) = \frac{1}{486}(17z^4 + 117z^3 + 218z^2 + 117z + 17)$
112	11011 <sub>3</sub>	$\tilde{P}_{112}(z) = \frac{1}{243}(11z^4 + 60z^3 + 101z^2 + 60z + 11)$
113	11012 <sub>3</sub>	$\tilde{P}_{113}(z) = \frac{1}{486}(5z^5 + 61z^4 + 177z^3 + 177z^2 + 61z + 5)$
115	11021 <sub>3</sub>	$\tilde{P}_{115}(z) = \frac{1}{486}(5z^5 + 59z^4 + 179z^3 + 179z^2 + 59z + 5)$
116	11022 <sub>3</sub>	$\tilde{P}_{116}(z) = \frac{1}{243}(10z^4 + 60z^3 + 103z^2 + 60z + 10)$
119	11102 <sub>3</sub>	$\tilde{P}_{119}(z) = \frac{1}{486}(6z^5 + 61z^4 + 176z^3 + 176z^2 + 61z + 6)$
121	11111 <sub>3</sub>	$\tilde{P}_{121}(z) = \frac{1}{486}(7z^5 + 65z^4 + 171z^3 + 171z^2 + 65z + 7)$
122*	11112 <sub>3</sub>	$\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$

**Table 2.** Several remarkable limit polynomials  $\tilde{P}_m(z)$  for  $m \leq 1094$ .

Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
<i>First occurrence of degree <math>d</math></i>		
1	1 <sub>3</sub>	$\tilde{P}_1(z) = \tilde{P}_3(z) = \tilde{P}_9(z) = \dots = \frac{1}{2}(1 + z)$
2*	2 <sub>3</sub>	$\tilde{P}_2(z) = \tilde{P}_6(z) = \dots = \frac{1}{6}(1 + 4z + z^2)$

Index $m$	Configuration	Polynomial $\tilde{P}_m(z)$
5*	12 <sub>3</sub>	$\tilde{P}_5(z) = \frac{1}{18}(z^3 + 8z^2 + 8z + 1)$
14*	112 <sub>3</sub>	$\tilde{P}_{14}(z) = \frac{1}{54}(z^4 + 13z^3 + 26z^2 + 13z + 1)$
41*	1112 <sub>3</sub>	$\tilde{P}_{41}(z) = \frac{1}{162}(z^5 + 19z^4 + 61z^3 + 61z^2 + 19z + 1)$
122*	11112 <sub>3</sub>	$\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$
365*	111112 <sub>3</sub>	$\tilde{P}_{365}(z) = \frac{1}{1458}(z^7 + 34z^6 + 211z^5 + 483z^4 + 483z^3 + 211z^2 + 34z + 1)$
1094*	1111112 <sub>3</sub>	$\tilde{P}_{1094}(z) = \frac{1}{4374}(z^8 + 43z^7 + 343z^6 + 1050z^5 + 1500z^4 + 1050z^3 + 343z^2 + 43z + 1)$
<i>Similar configurations</i>		
122	11112 <sub>3</sub>	$\tilde{P}_{122}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$
124	11121 <sub>3</sub>	$\tilde{P}_{124}(z) = \frac{1}{486}(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1)$
130	11211 <sub>3</sub>	$\tilde{P}_{130}(z) = \frac{1}{486}(z^6 + 22z^5 + 120z^4 + 200z^3 + 120z^2 + 22z + 1)$
148	12111 <sub>3</sub>	$\tilde{P}_{148}(z) = \frac{1}{486}(z^6 + 23z^5 + 119z^4 + 200z^3 + 119z^2 + 23z + 1)$
202	21111 <sub>3</sub>	$\tilde{P}_{202}(z) = \frac{1}{486}(z^6 + 26z^5 + 120z^4 + 192z^3 + 120z^2 + 26z + 1)$
<i>Irreducible up to a root <math>(-1)</math> cubic polynomials</i>		
91	10101 <sub>3</sub>	$\tilde{P}_{91}(z) = \frac{1}{486}(56z^3 + 187z^2 + 187z + 56)$
253	100101 <sub>3</sub>	$\tilde{P}_{253}(z) = \frac{1}{1458}(173z^3 + 556z^2 + 556z + 173)$
739	1000101 <sub>3</sub>	$\tilde{P}_{739}(z) = \frac{1}{4374}(524z^3 + 1663z^2 + 1663z + 524)$
757	1001001 <sub>3</sub>	$\tilde{P}_{757}(z) = \frac{1}{4374}(533z^3 + 1654z^2 + 1654z + 533)$

**Table 3.** Non-irreducible polynomials  $\tilde{P}_m(z)$  for  $m \leq 122$  (up to a root  $-1$ ).

Index $m$	Configuration	Factorization of $\tilde{P}_m(z)$
4	$11_3$	$\tilde{P}_4(z) = \frac{1}{9}(2+z)(1+2z)$
8	$22_3$	$\tilde{P}_8(z) = \frac{1}{9}(2+z)(1+2z)$
28	$1001_3$	$\tilde{P}_{28}(z) = \frac{1}{81}(5+4z)(4+5z)$
40	$1111_3$	$\tilde{P}_{40}(z) = \frac{1}{81}(3+5z+z^2)(1+5z+3z^2)$
52	$1221_3$	$\tilde{P}_{52}(z) = \frac{1}{81}(2+6z+z^2)(1+6z+2z^2)$
56	$2002_3$	$\tilde{P}_{56}(z) = \frac{1}{81}(2+6z+z^2)(1+6z+2z^2)$
68	$2112_3$	$\tilde{P}_{68}(z) = \frac{1}{81}(3+5z+z^2)(1+5z+3z^2)$
80	$2222_3$	$\tilde{P}_{80}(z) = \frac{1}{9}(5+4z)(4+5z)$
244	$111111_3$	$\tilde{P}_{244}(z) = \frac{1}{729}(14+13z)(13+14z)$

## VI. REFERENCES

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- [Cha69] R.V. Chacon. Weakly mixing transformations which are not strongly mixing. *Proc. Amer. Math. Soc.*, 22:559–562, 1969.
- [Dan04] A.I. Danilenko. Infinite rank-one actions and nonsingular chacon transformations. *Illinois. J. Math.*, 48(3):769–786, 2004.
- [dJ78] A. del Junco. A simple measure-preserving transformation with trivial centralizer. *Pacific J. Math.*, 79:357–362, 1978.
- [dJL92] A. del Junco and M. Lemanczyk. Generic spectral properties of measure preserving maps and applications. *Proc. Amer. Math. Soc.*, 115:725–736, 1992.
- [dJRS80] A. del Junco, M. Rahe, and L. Swanson. Chacon’s automorphism has minimal self-joinings. *J. Anal. Math.*, 37:276–284, 1980.
- [Fri70] N.A. Friedman. *Introduction to ergodic theory*. Van Nostrand, 1970.
- [Goo99] G. Goodson. A survey of recent results in the spectral theory of ergodic dynamical systems. *J. Dynam. Control Systems*, 5:173–226, 1999.
- [KT07] A. Katok and J.-P. Thouvenot. *Spectral theory and combinatorial constructions. Handbook on dynamical systems*, volume 1B. Elsevier, Amsterdam, 2007.
- [Lem09] M. Lemanczyk. *Spectral Theory of Dynamical Systems, Encyclopedia of Complexity and System Science*. Springer Verlag, 2009.
- [Ose69] V.I. Oseledets. An automorphism with simple continuous spectrum without the group property. *Russian Math. Notes*, 5:196–198, 1969.
- [PR] A.A. Prihod’ko and V.V. Ryzhikov. Disjointness of the convolutions for Chacon’s automorphism. *Colloquium Mathematicum*, 84/85.

[Ste87] A.M. Stepin. Spectral properties of generic dynamical systems. *Math. USSR-Izv.*, 29:159–192, 1987.

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